The Virtual Element Method

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A few words on VEM

- The Virtual Element Method was born an extension of classical Finite Element Method to polygons and polyhedra.
- It has its roots in the Mimetic Finite Difference Method.
- First paper appeared in 2013:
  L. Beirão da Veiga, F. Brezzi, A. Cangiani, G. Manzini, L.D. Marini, A. Russo: *Basic principles of Virtual Element Methods*, Math. Models Methods Appl. Sci. 23, (2013), 199-214.
- Has attracted much attention (268 citations in Google Scholar).
- VEM ideas have many applications.
- Many people are working on it (also independently of us).

Program

- The Finite Element Method in a nutshell
- Basics of the Virtual Element Method
- Numerical experiments in MATLAB
- Advanced topics (?)

Lax-Milgram Lemma

Let V be a Hilbert space, and let $a(\cdot,\cdot)$ be a bilinear form on $V \times V$:

$$a(\cdot,\cdot) : V \times V \longrightarrow \mathbb{R}$$

which is *continuous*, i.e. there exists a constant M such that

$$a(u,v) \leq M \|u\|_V \|v\|_V \text{ for all } u \text{ and } v \text{ in } V$$

and *coercive*, i.e. there exists a constant α such that

$$a(v,v) \geq \alpha \|v\|_V^2.$$

In addition, let $F : V \longrightarrow \mathbb{R}$ be a continuous linear form on V, i.e. an element of the dual V'.

Lax-Milgram Lemma

Then we consider the abstract variational problem:

$$\begin{cases} \text{find } u \in V \text{ such that} \\ a(u,v) = F(v) \quad \text{for all } v \in V \end{cases} \tag{1}$$

Lax-Milgram Lemma (1954) guarantees that under the previous assumptions on $a(\cdot,\cdot)$ and on $F(\cdot)$ problem (1) has a unique solution which depends continuously on the data.

Proof:

- $a(\cdot,\cdot)$ symmetric: Riesz representation theorem - ($a(\cdot,\cdot)$ is a scalar product on V);
- general case: Banach fixed point theorem.

Lax-Milgram Lemma

Some insight into the theorem can be obtained with the following elementary considerations.

Consider the (linear and continuous) operator $A : V \longrightarrow V'$ defined by

$$(Au)(v) := a(u, v).$$

Then problem (1) can be restated as an equation in V':

$$\text{find } u \text{ in } V \quad \text{such that} \quad Au = F \text{ in V'}$$

The coercivity assumption on $a(\cdot, \cdot)$ assures that the operator A is injective:

$$Au = 0 \implies (Au)(u) = a(u, u) = 0$$

but by coercivity

$$0 = a(u, u) \geq \alpha \|u\|_V^2 \implies u = 0.$$

Lax-Milgram Lemma

Hence $A : V \longrightarrow V'$ is injective.

If V is finite dimensional, then also V' is finite dimensional and furthermore

$$\dim V = \dim V'.$$

Hence by the so-called rank–nullity theorem, A is also surjective so the equation $Au = F$ has a unique solution for all $F \in V'$.

Example - Poisson equation

Let Ω be a domain in $\mathbb{R}^n$ with a "reasonable boundary" (Lipschitz is OK), and $f \in L^2(\Omega)$. Then we can take:

$$V := H_0^1(\Omega), \quad a(u,v) := \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x, \quad F(v) = \int_\Omega f\, v \, \mathrm{d}x$$

The abstract variational problem becomes:

$$\begin{cases} \text{find } u \in H_0^1(\Omega) \text{ such that} \\ \displaystyle\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = \int_\Omega f\, v \, \mathrm{d}x \quad \text{for all } v \in H_0^1(\Omega). \end{cases}$$

This is the weak form of Poisson equation with homogenous Dirichlet boundary conditions:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Example - Poisson equation with variable diffusion

Let $\kappa : \Omega \longrightarrow \mathbb{R}$ be a "reasonable" function (L^∞ is OK) which is strictly positive, i.e.

$$\kappa(x) \geq \kappa_0 > 0.$$

Then we can take:

$$a(u, v) := \int_\Omega \kappa \nabla u \cdot \nabla v \, \mathrm{d}x$$

The abstract variational problem now becomes:

$$\begin{cases} \text{find } u \in H_0^1(\Omega) \text{ such that} \\ \displaystyle\int_\Omega \kappa \nabla u \cdot \nabla v \, \mathrm{d}x = \int_\Omega f \, v \, \mathrm{d}x \quad \text{for all } v \in H_0^1(\Omega) \end{cases}$$

This is the weak form of the following partial differential equation:

$$\begin{cases} -\operatorname{div}(\kappa \nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Galerkin approximation

Let now V_h be a finite dimensional subspace of V (a closed subspace would also work).

By restricting $a(\cdot,\cdot)$ and $F(\cdot)$ to V_h, we can apply Lax-Milgram Lemma to the same problem we had before restricted to V_h:

$$\begin{cases} \text{find } u_h \in V_h \text{ such that} \\ a(u_h, v_h) = F(v_h) \quad \text{for all } v_h \in V_h \end{cases} \tag{2}$$

Problem (2) has a unique solution u_h because all the hypotheses on $a(\cdot,\cdot)$ and $F(\cdot)$ are verified on V_h, since it is a subspace of V.

The function $u_h \in V_h \subset V$ is the Galerkin approximation of $u \in V$.

Properties of the Galerkin approximation u_h

In Numerical Analysis, a crucial concept is "consistency", i.e. "to what extent the exact solution satisfies the approximate equation".

For the FEM, we have that the exact solution u satisfies **exactly** the approximate equation. In fact, we have

$$a(u, v) = F(v) \quad \text{for all } v \in V$$

and since $V_h \subset V$ we also have trivially

$$a(u, v_h) = F(v_h) \quad \text{for all } v_h \in V_h.$$

Exact consistency has as a consequence a couple of properties that are of the utmost importance for the Finite Element Method.

Properties of the Galerkin approximation u_h

- "Galerkin orthogonality":

$$a(u - u_h, v_h) = 0 \quad \text{for all } v_h \in V_h$$

Proof: subtract $a(u, v_h) = F(v_h)$ from $a(u_h, v_h) = F(v_h)$.

- Céa's Lemma:

$$\|u - u_h\|_V \leq \frac{M}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|_V$$

Proof:

$\alpha \|u - u_h\|_V^2 \leq a(u - u_h, u - u_h) = a(u - u_h, u - v_h) \leq M \|u - u_h\|_V \|u - v_h\|_V$

- "Patch Test":

$$u \in V_h \quad \text{implies} \quad u_h = u$$

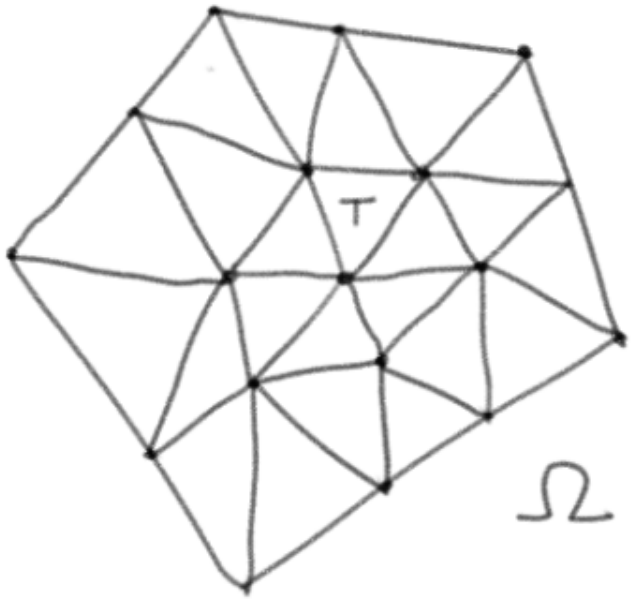
Proof: Céa's Lemma.

Construction of V_h for the Poisson Problem

In view of the approximation of the Poisson equation, we take $V = H_0^1(\Omega)$ and we try to construct a finite dimensional space $V_h \subset H_0^1(\Omega)$ which is "close enough" to the space $H_0^1(\Omega)$.

Thanks to Céa's Lemma, this will ensure that u_h is "close enough" to u.

We stick on the two-dimensional case. We assume that Ω is a polygon and we take a simplicial decomposition of Ω in triangles $\mathcal{T}_h = \{T\}$.



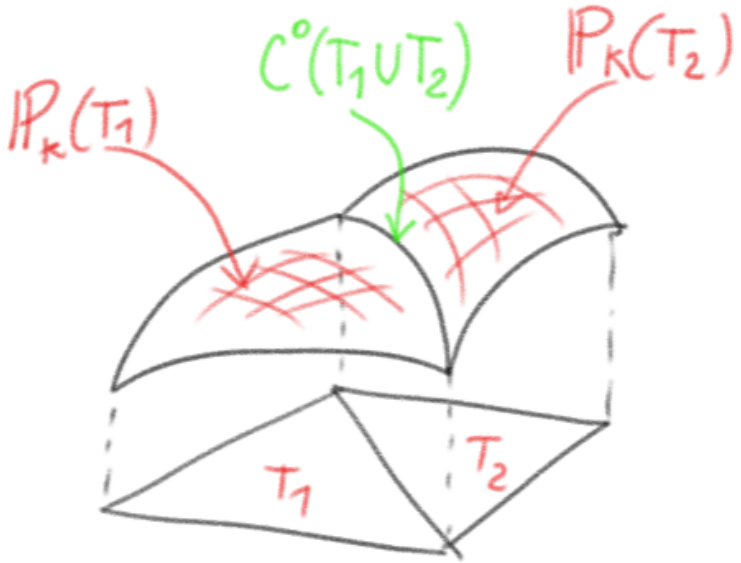
Construction of V_h for the Poisson Problem

We consider the following linear space V_h:

$$V_h := \{v_h \in C^0(\bar{\Omega}) \cap H_0^1(\Omega) \text{ such that } v_{h|T} \in \mathbb{P}_k(T)\}$$

Hence a function $v_h \in V_h$ has the following properties:

- when restricted to a triangle is a polynomial of degree k;
- is continuous across two adjacent triangles;



- is zero on the boundary of Ω.

Construction of V_h for the Poisson Problem

Remark: In the definition of V_h the space $C^0(\bar{\Omega}) \cap H_0^1(\Omega)$ can be substituted simply by $H_0^1(\Omega)$ since, for regular functions (here we have polynomials), H^1 continuity across triangles implies C^0 continuity.

The subscript h in V_h stands for the maximum diameter of the triangles in the decomposition $\mathcal{T}_h$:

$$h = \max_{T \in \mathcal{T}_h} \{\operatorname{diam} T\}.$$

Under suitable assumptions we have the following convergence results.
Let $\{\mathcal{T}_h\}_h$ be a "regular" family of decompositions of Ω with decreasing h.
Then:

$$|u - u_h|_{1,\Omega} \leq C h^k |u|_{k+1,\Omega}$$

and

$$\|u - u_h\|_{0,\Omega} \leq C h^{k+1} |u|_{k+1,\Omega}$$

So far so good.

Construction of V_h for the Poisson Problem

In practice, to find the function u_h one should define a basis of V_h and then compute everything in this basis.

In order to understand how to construct a basis for V_h, we start to describe the local space $\mathbb{P}_k(T)$ on a triangle T.

Let π_k be the dimension of the space of polynomials on T, i.e. the dimension of the space of polynomials in two variables:

$$\pi_k = \dim \mathbb{P}_k(T) = \frac{(k+1)(k+2)}{2}$$

Local Degrees of Freedom

We start by defining the concept of degree of freedom, and then we will define a local basis.

A degree of freedom is simply a linear functional defined on $\mathbb{P}_k(T)$.

A classical example of degree of freedom is the value at a point Q of the triangle:

$$\mathrm{dof}_Q(p_k) = p_k(Q), \quad p_k \in \mathbb{P}_k(T).$$

Clearly, $\mathrm{dof}_Q \in \mathbb{P}_k(T)'$.

We can take also other functionals different from the pointwise value, for instance the integral against a fixed function ψ defined on T:

$$\mathrm{dof}_\psi(p_k) = \int_T p_k \, \psi \, \mathrm{d}x.$$

Local Degrees of Freedom

A set of degrees of freedom

$$\mathcal{D} = \{\mathrm{dof}_i,\ i = 1, \ldots, N^{\mathrm{dof}}\}$$

is called unisolvent on $\mathbb{P}_k(T)$ if the linear functionals $\{\mathrm{dof}_i\}$ are linearly independent.

In other words, the linear map $D : \mathbb{P}_k(T) \longrightarrow \mathbb{R}^{N^{\mathrm{dof}}}$ defined by

$$i\text{-th component of } Dp_k := \mathrm{dof}_i(p_k)$$

should be an isomorphism.

In practical cases, unisolvence can often be proved more easily by checking the following two conditions:

- the number of dof's is equal to the dimension, i.e. $N^{\mathrm{dof}} = \pi_k$;
- $\boxed{\mathrm{dof}_i(p_k) = 0 \text{ for } i = 1, \ldots, N^{\mathrm{dof}}}$ implies $\boxed{p_k \equiv 0}$

Local Degrees of Freedom

Given a unisolvent set of degrees of freedom, we can immediately define a basis for $\mathbb{P}_k(T)$.

For $j = 1, \dots \pi_k$ we define $\varphi_j \in \mathbb{P}_k(T)$ as the (unique) polynomial satisfying

$$\mathrm{dof}_i(\varphi_j) = \delta_{ij}.$$

We have the Lagrange representation:

$$p_k = \sum_{i=1}^{\pi_k} \mathrm{dof}_i(p_k)\varphi_i$$

Proof: write $p_k = \sum_{j=1}^{\pi_k} c_i \varphi_j$ and then apply the operator dof_i to both terms to obtain $c_i = \mathrm{dof}_i(p_k)$.

Local Degrees of Freedom

For classical Lagrange Finite Element, we consider π_k distinct points N_i in the triangle T (the nodes) and we consider the set of degrees of freedom

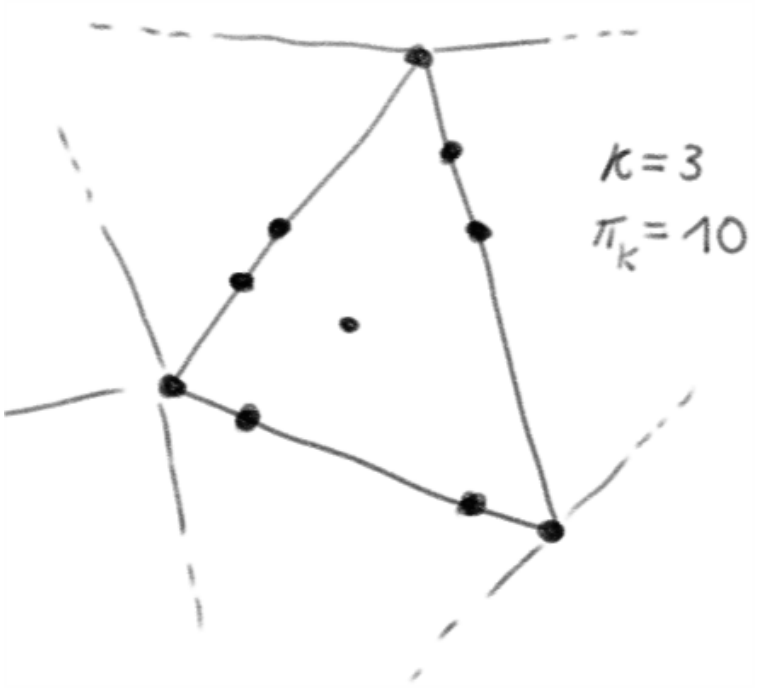
$$\mathcal{D} = \{\mathrm{dof}_{N_i},\ i = 1, \ldots, \pi_k\}$$

where $\mathrm{dof}_{N_i}(p_k) = p_k(N_i)$. The corresponding polynomial basis is made of polynomials φ_i such that $\varphi_i(N_j) = \delta_{ij}$.

The location of the nodes N_i is chosen in such a way as the continuity across elements is automatically guaranteed. Hence as nodes we take:

- the vertices;
- $(k-1)$ points on each edge (often are taken equispaced, but any other choice would work);
- $\pi_k - 3 - 3(k-1) = \pi_{k-3}$ points inside the triangle (anywhere).

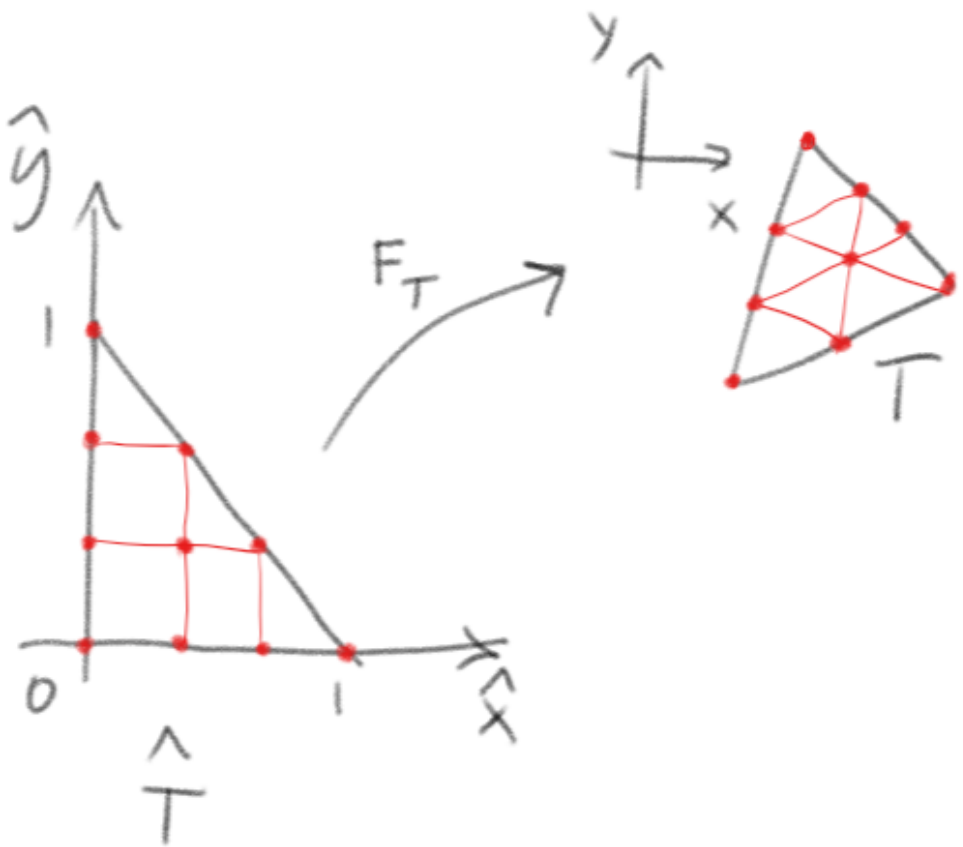
Odd nodes



Caveat: we are not considering boundary condition very seriously. Of course, if a node is on a Dirichlet boundary, then it is no more a degree of freedom since the value of u_h there is zero (or in general determined by the boundary condition).

Nodes coming from the reference element

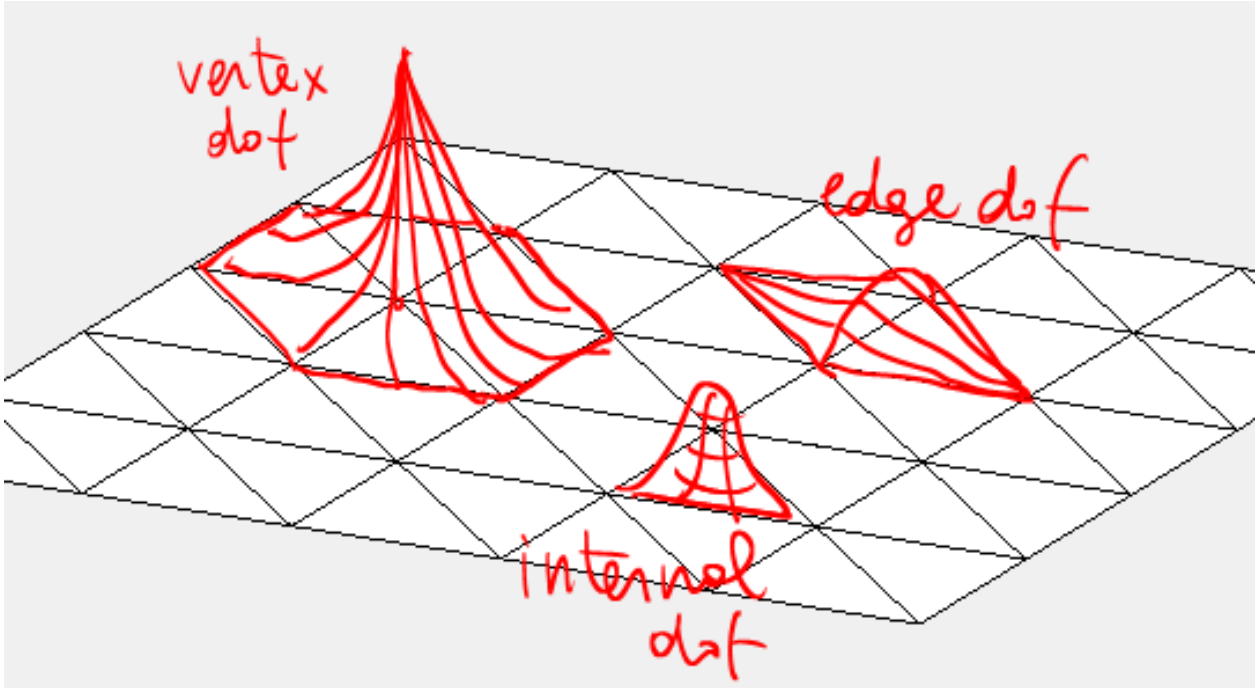
To take advantage of the reference element technology, usually the nodes in the current triangles are the affine image of the nodes in the reference triangle, hence they are in "fixed" locations.



Global basis functions

Local degrees of freedom are put all together and give the set of Global Degrees of Freedom.

Consequently, local basis functions are glued together in order to obtain global basis functions



Computation of u_h

Once we have a basis of the space V_h whose elements are still called φ_i, where this time i varies among all N nodes of the triangulation, we proceed in the following way.

First, we take $v_h = \varphi_i$ in the variational equation which becomes:

$$a(u_h, \varphi_i) = F(\varphi_i), \quad i = 1, \dots, N$$

Then we write u_h as a linear combination of the φ_j's:

$$u_h = \sum_{j=1}^{N} u_j \varphi_j$$

and we end up with a system of linear equations:

$$\sum_{j=1}^{N} a(\varphi_j, \varphi_i) u_j = F(\varphi_i)$$

Local and global stiffness matrix

The so-called stiffness matrix $a(\varphi_j, \varphi_i) = \int_\Omega \kappa \nabla\varphi_j \cdot \nabla\varphi_i \, \mathrm{d}x$ can be distributed among the elements as

$$a(\varphi_j, \varphi_i) = \sum_{T \in \mathcal{T}_h} a^T(\varphi_j, \varphi_i)$$

where

$$a^T(\varphi_j, \varphi_i) = \int_T \kappa \nabla\varphi_j \cdot \nabla\varphi_i \, \mathrm{d}x$$

and by construction $a^T(\varphi_j, \varphi_i) = 0$ unless both nodes i and j belongs to triangle T.

Hence the local matrix $a^T(\varphi_j, \varphi_i)$ "is" a $\pi_k \times \pi_k$ matrix with i and j varying only among the π_k nodes of the triangle T. This procedure is called "assembling".

Local and global stiffness matrix

For the right-hand-side we do the same:

$$F(\varphi_i) = \sum_{T \in \mathcal{T}_h} F^T(\varphi_i) \quad \text{with } F^T(\varphi_i) = \int_T f \varphi_i \, \mathrm{d}x.$$

Hence, if nodes are chosen properly so that everything glues together nicely, we only have to compute the so-called local stiffness matrix:

$$a^T(\varphi_j, \varphi_i) = \int_T \kappa \nabla \varphi_j \cdot \nabla \varphi_i \, \mathrm{d}x$$

and the local load term:

$$F^T(\varphi_i) = \int_T f \varphi_i \, \mathrm{d}x$$

where with a small abuse of notation the indices i and j here are "local to the triangle T" and run from 1 to $\pi_k = \dim \mathbb{P}_k(T)$.

Numerical integration

Any deviation from the procedure described above is called a "variational crime" (copyright G. Strang). The easiest variational crime to understand is the numerical integration:

the terms $a^T(\varphi_j, \varphi_i)$ and $F^T(\varphi_i)$ cannot in general be computed exactly, but have to be approximated by a quadrature formula on T:

$$a^T(\varphi_j, \varphi_i) \approx a_h^T(\varphi_j, \varphi_i), \quad F^T(\varphi_i) \approx F_h^T(\varphi_i)$$

giving rise to an approximate global stiffness matrix and load term:

$$a_h(\varphi_j, \varphi_i) = \sum_{T \in \mathcal{T}_h} a_h^T(\varphi_j, \varphi_i), \quad F_h(\varphi_i) = \sum_{T \in \mathcal{T}_h} F_h^T(\varphi_i).$$

Hence in the Galerkin approximation we need to use more h's:

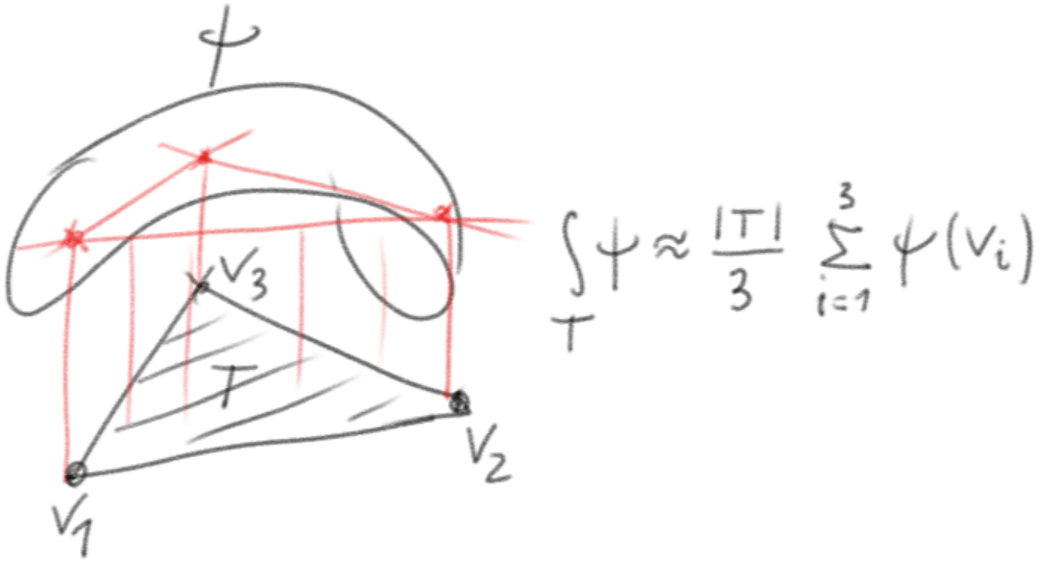
not $a(u_h, v_h) = F(v_h)$ but $a_h(u_h, v_h) = F_h(v_h)$ for all $v_h V_h$.

Numerical integration

The error introduced by the numerical integration should be so small to not upset the convergence error.

It can be shown that it is enough to consider a quadrature formula with degree of precision (at least) $\boxed{2k-1}$.

Such formulae do exist for triangles (unfortunately they are not Gauss-like) and since every triangle can be mapped affinely to any other triangle, it is enough to compute them once and for all on a "reference" triangle.



What is the Virtual Element Method?

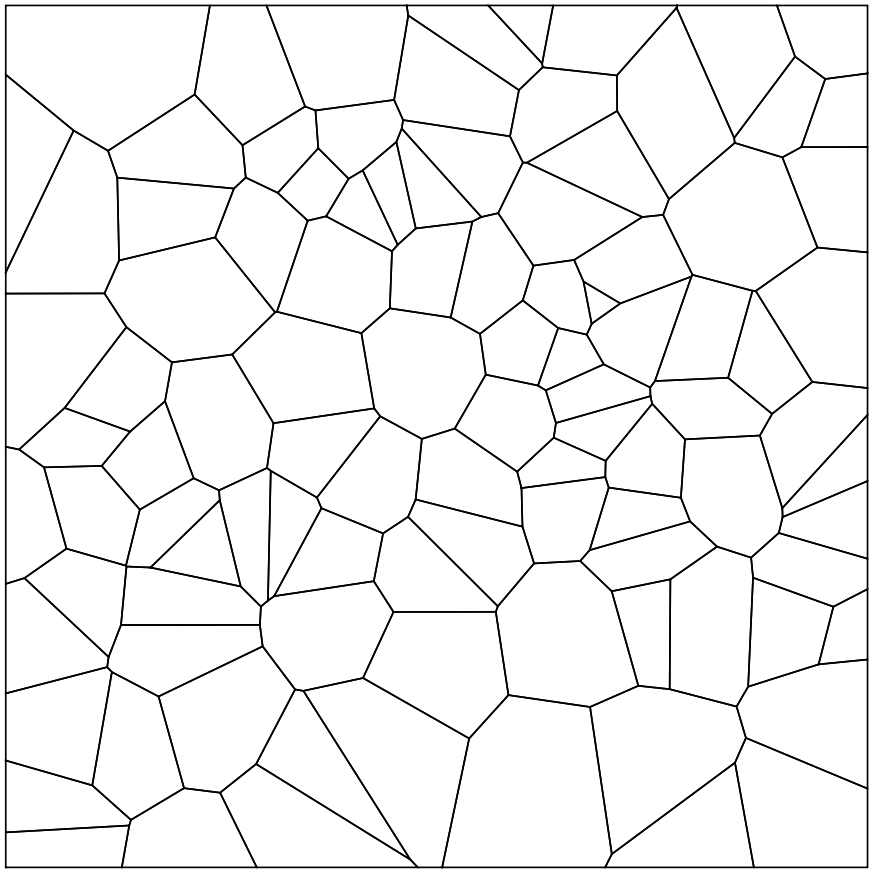
In short, the Virtual Element Method is an extension of the Finite Element Method to polygons, but it is not restricted to this. In more complicated situations (mixed methods, higher regularity, three dimensions), with the ideas of VEM we can define new finite elements also on triangles and tetrahedra.

Why polygons?

- Greater geometrical flexibility;
- Interaction with FEM: derefinement;
- Interaction with FEM: hanging nodes;
- Treatment of curved boundary.
- The main reason is that the theory is very elegant. We hope it will also be useful...

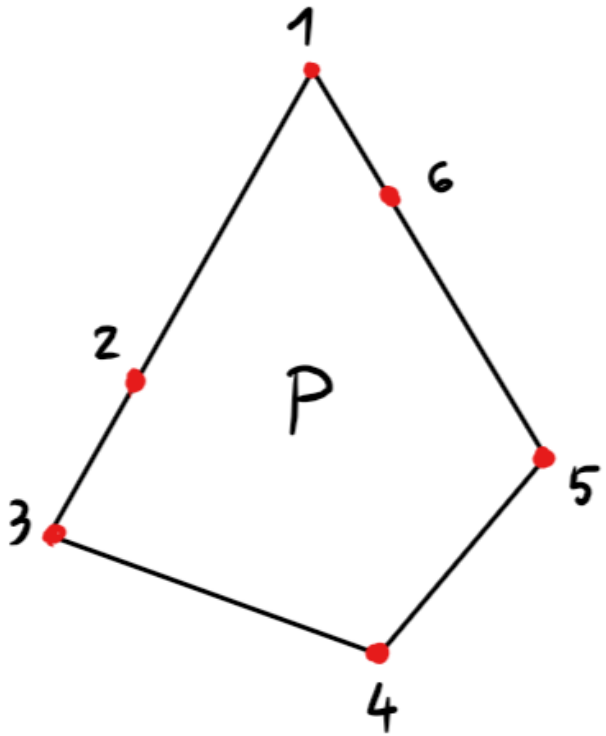
Polynomial decomposition of the domain Ω

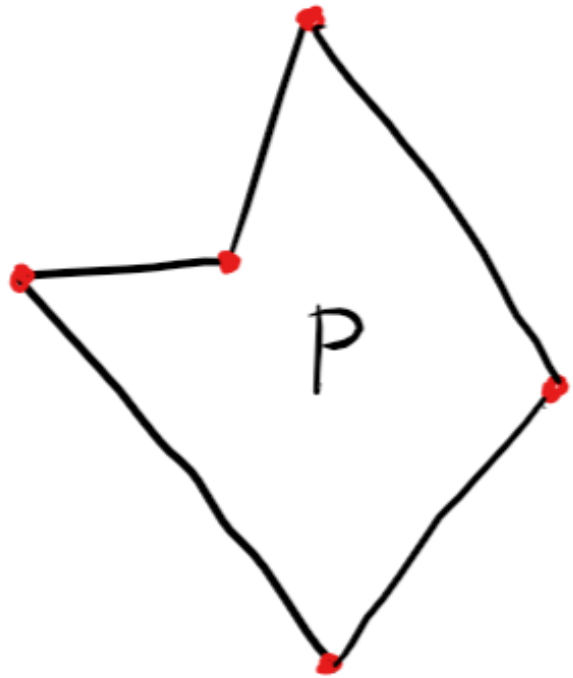
We assume that the domain Ω has been partitioned in a collection $\mathcal{P}_h$ of polygons:



Polynomial decomposition of the domain Ω

We allow "hanging nodes", i.e. consecutive edges on the same line, and also non-convex polygons:





This is a Hexagon!

The Model Problem

For the time being, we stick on Laplace equation. Later on we'll see how to extend the method to a general elliptic operator.

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

The construction of the Virtual Element Method will follow very closely what we did for the classical (conforming) Finite Element Method.

Basically, we construct local spaces on each polygon and then we glue them together, exactly as for FEM.

A mantra: our local VEM spaces MUST contain polynomials of degree less then or equal to a certain degree k. This is mandatory for convergence!!!

The local finite element space $V_k(P)$

Let P be a polygon. Mimicking the classical Lagrange Finite Elements for triangles, we would like to define a finite element space $V_k(P)$ on P such that:

- $V_k(P)$ contains the space $\mathbb{P}_k(P)$ of polynomials of degree less than or equal to k (plus - possibly - other non-polynomial functions);
- a function in $V_k(P)$ restricted to an edge e is in $\mathbb{P}_k(e)$, so that if two polygons P and P' have an edge in common, the two spaces $V_k(P)$ and $V_k(P')$ must "glue" in $C^0(P \cup P')$ (compatibility with FEM);
- given functions $u_h, v_h \in V_k(P)$, I can "compute" the bilinear form $a(u_h, v_h)$ and the load term $F(v_h)$ (or at least good approximations).

The space $V_k(P)$

We start with a trivial remark:

$$p_k \in \mathbb{P}_k(P) \implies \Delta p_k \in \mathbb{P}_{k-2}(P).$$

Since we want that our space contains the polynomials of degree k, we consider the following space:

$$V_k(P) := \{v_h \in C^0(P) \text{ such that: } v_{h|e} \in \mathbb{P}_k(e),\ \Delta v_h \in \mathbb{P}_{k-2}(P)\}$$

Is the space $V_k(P)$ OK?

The space $V_k(P)$

Is the space $V_k(P)$ OK?

YES: it contains polynomials of degree up to k and is conformal, so (under some regularity assumptions on the polygonal mesh - to be discussed later) order k approximation in H^1 is guaranteed....
$\Longrightarrow$ Polygonal FEM, Wachspress or barycentric coordinates...

BUT: the functions in $V_k(P)$ are not known explicitly! They are defined through the solution of a PDE in the element! ($\longrightarrow$ polygonal FEM)

$$\int_P \nabla\varphi_j \cdot \nabla\varphi_i = ???$$

Scaled monomials: a basis for $\mathbb{P}_k(P)$

Let (x_P, y_P) be the centroid of P and h_P its diameter. If $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ is a multiindex we define the scaled monomials of degree $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2$:

$$m_{\boldsymbol{\alpha}}(x, y) := \left(\frac{x - x_P}{h_P}\right)^{\alpha_1} \left(\frac{y - y_P}{h_P}\right)^{\alpha_2}.$$

Clearly, the set

$$\mathcal{M}_k := \{m_{\boldsymbol{\alpha}}, \text{ with } 0 \leq |\boldsymbol{\alpha}| \leq k\}$$

is a basis for $\mathbb{P}_k(P)$.

Degrees of freedom in $V_k(P)$

If the polygon E has N_e edges (and $N_V = N_e$ vertices), it is clear that

$$\dim V_k(P) = N_V + (k-1)\, N_e + \dim \mathbb{P}_{k-2}(P) = k\, N_e + \pi_{k-2}$$

As degrees of freedom in $V_k(P)$, we choose:

- the value of v_h at the vertices and at $k-1$ points on each edge;
- the (scaled) moments $\boxed{\frac{1}{|P|}\int_P v_h m_{\boldsymbol{\alpha}}\,\mathrm{d}x}$ for $0 \le |\boldsymbol{\alpha}| \le k-2$.

It can be easily shown that the degrees of freedom above are *unisolvent* in $V_k(P)$.

Degrees of freedom in $V_k(P)$

Proof.

- Clearly the number of (candidate) dofs equals the dimension of $V_k(P)$.
- Hence, it is enough to show that if the dofs of a function $v_h \in V_k(P)$ are all zero, then the function v_h is identically zero.
- First of all, it is clear that v_h is identically zero on the boundary of P.
- Suppose then that $v_h \in V_k(P)$ is zero on ∂P and

$$\int_P v_h \, m_{\boldsymbol{\alpha}} \, \mathrm{d}x = 0 \quad \text{for } 0 \leq |\boldsymbol{\alpha}| \leq k-2.$$

We show that $v_h \equiv 0$. For, integrating by parts,

$$\int_P |\nabla v_h|^2 \, \mathrm{d}x = -\int_P \Delta v_h \, v_h \, \mathrm{d}x + \int_{\partial P} \frac{\partial v_h}{\partial n} \, v_h \, \mathrm{d}s$$

Degrees of freedom in $V_k(P)$

We show now that

$$\int_P |\nabla v_h|^2 \,\mathrm{d}x = -\int_P \Delta v_h \, v_h \,\mathrm{d}x + \int_{\partial P} \frac{\partial v_h}{\partial n} \, v_h \,\mathrm{d}s = 0.$$

Infact:

- $\int_P \Delta v_h \, v_h \,\mathrm{d}x = 0$ because Δv_h is a polynomial of degree $k-2$ and by hypothesis all moments of v_h up to order $k-2$ are zero;
- $\int_{\partial P} \frac{\partial v_h}{\partial n} \, v_h \,\mathrm{d}s = 0$ since $v_h = 0$ on the boundary of P.

Hence ∇v_h is zero so v_h is constant, and since $v_h = 0$ on the boundary of P, we conclude that $v_h \equiv 0$. □

Comparison with Lagrange FE on triangles

Remark: also in the case when P is a triangle, for $k \geq 2$ we always have

$$\boxed{V_k(P) \supsetneq \mathbb{P}_k(P)}$$

- If P is a triangle, we have $V_1(P) \equiv \mathbb{P}_1(P)$.
- If P is a triangle and $k = 2$, we have

$$V_2(P) = \mathbb{P}_2(P) \oplus \{\text{one bubble}\}$$

The bubble in general is not a polynomial.

$\Longrightarrow$ There is a general technique (serendipity VEM) to sistematically reduce the number of internal degrees of freedom without losing accuracy.

Basis functions in $V_k(P)$

For $i = 1, \dots, N^{\text{dof}} := \dim V_k(P)$ we define φ_i as the function in $V_k(P)$ such that

$$\text{dof}_j(\varphi_i) = j\text{-th degree of freedom of } \varphi_i = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

We have the usual Lagrange-type expansion

$$\boxed{v_h = \sum_{i=1}^{N^{\text{dof}}} \text{dof}_i(v_h)\, \varphi_i.}$$

$\Longrightarrow$ examples of "true" basis functions.

Meaning of "I can compute"

Now we need to **extract some information from our VEM space**. We start by making more precise this concept.

If $\mathcal{A}v_h$ is some quantity depending on v_h (for instance a projection onto polynomials), in what follows the precise meaning of the statement

I can compute $\mathcal{A}v_h$

is:

given the array $\mathrm{dof}_i(v_h)$, I can directly obtain $\mathcal{A}v_h$ by integration of polynomials on P only (and maybe solving a linear system :-)

THe L^2 projector onto $\mathbb{P}_{k-2}(P)$

The choice of the scaled moments $\frac{1}{|P|}\int_P v_h m_{\boldsymbol{\alpha}}$ among the degrees of freedom implies that, starting from the degrees of freedom of v_h, I can trivially compute

$$\Pi^0_{k-2} v_h := L^2 \text{ projection of } v_h \text{ onto } \mathbb{P}_{k-2}(P).$$

Hence

$$\Pi^0_{k-2}\varphi_i \text{ is a computable polynomial of degree } k-2$$

Proof. Write $\Pi^0_{k-2}\varphi_i = \sum_{|\boldsymbol{\alpha}|=1}^{k} c_{\boldsymbol{\alpha}} m_{\boldsymbol{\alpha}}$; multiply by $m_{\boldsymbol{\beta}}$ and integrate over P:

$$\sum_{|\boldsymbol{\alpha}|=1}^{k} c_{\boldsymbol{\alpha}} \frac{1}{|P|}\int_P m_{\boldsymbol{\alpha}} m_{\boldsymbol{\beta}}\,\mathrm{d}x = \frac{1}{|P|}\int_P \Pi^0_{k-2}\varphi_i m_{\boldsymbol{\beta}}\,\mathrm{d}x$$

$$= \frac{1}{|P|}\int_P \varphi_i m_{\boldsymbol{\beta}}\,\mathrm{d}x = \begin{cases} 0 & \text{if } i \text{ is a boundary dof} \\ 0 \text{ or } 1 & \text{if } i \text{ is an internal dof} \end{cases}$$

and we have to integrate the polynomial $m_{\boldsymbol{\alpha}} m_{\boldsymbol{\beta}}$ on P and solve a linear system.

The Π_k^∇ projector onto $\mathbb{P}_k(P)$

We can define a projector $\Pi_k^\nabla : V_k(E) \to \mathbb{P}_k(P)$ which is computable.

The projector Π_k^∇ is orthogonal with respect to the energy scalar product in $H^1(P)$:

$$\int_P \nabla\left[\Pi_k^\nabla v_h\right] \cdot \nabla p_k \,\mathrm{d}x = \int_P \nabla v_h \cdot \nabla p_k \,\mathrm{d}x \quad \text{for all } p_k \in \mathbb{P}_k(P)$$

$$\int_{\partial P} \Pi_k^\nabla v_h \,\mathrm{d}x = \int_{\partial P} v_h \,\mathrm{d}x \quad \text{to fix the constant function}$$

$\Pi_k^\nabla \varphi_i$ is computable since:

$$\int_P \nabla\varphi_i \cdot \nabla p_k \,\mathrm{d}x = -\int_P \varphi_i \,\Delta p_k \,\mathrm{d}x + \int_{\partial P} \varphi_i \,\frac{\partial p_k}{\partial n} \,\mathrm{d}s$$

and the result follows observing that $\Delta p_k \in \mathbb{P}_{k-2}(P)$.

VEM approximation of the Poisson equation

We define the global space $V_h \subset H_0^1(\Omega)$ just by gluing together the local spaces $V_k(P)$ along the edges, as done in classical Lagrange FEM.

We consider now the Poisson equation:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

and its classical variational formulation:

find $u \in H_0^1(\Omega)$ such that $\displaystyle\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = \int_\Omega f\, v \, \mathrm{d}x$ for all $v \in H_0^1(\Omega)$.

We simply take its Galerkin approximation on the space $V_h \subset H_0^1(\Omega)$:

find $u \in V_h$ such that $\displaystyle\int_\Omega \nabla u_h \cdot \nabla v_h \, \mathrm{d}x = \int_\Omega f\, v_h \, \mathrm{d}x$ for all $v_h \in V_h$.

Consistency and Stability

We take $v_h = \varphi_i$, we write $u_h = \sum u_j \varphi_j$, we split the integral over the polygons P and we are left with the problem of computing the local stiffness matrices and the load vectors:

$$a^P(\varphi_j, \varphi_i) = \int_P \nabla\varphi_j \cdot \nabla\varphi_i \,\mathrm{d}x, \qquad F^P(\varphi_i) = \int_\Omega f\,\varphi_i \,\mathrm{d}x.$$

Unfortunately, these integrals are not computable. We need an idea.

We make the following crucial observation:

$$\int_P \nabla u_h \cdot \nabla v_h \,\mathrm{d}x = \underbrace{\int_P \nabla\Pi_k^\nabla u_h \cdot \nabla\Pi_k^\nabla v_h \,\mathrm{d}x}_{\text{consistency (computable! ☺)}} + \underbrace{\int_P \nabla(I-\Pi_k^\nabla)u_h \cdot \nabla(I-\Pi_k^\nabla)v_h \,\mathrm{d}x}_{\text{stability (NOT computable... ☹)}}$$

Consistency and Stability

Proof. We have the decomposition

$$w_h = \Pi_k^\nabla w_h + (I - \Pi_k^\nabla) w_h$$

hence it is enough to show that the "mixed" terms are zero, for instance:

$$\int_P \nabla (I - \Pi_k^\nabla) u_h \cdot \nabla \Pi_k^\nabla v_h \, \mathrm{d}x = 0$$

i.e.

$$\int_P \nabla u_h \cdot \nabla \Pi_k^\nabla v_h \, \mathrm{d}x = \int_P \nabla \Pi_k^\nabla u_h \cdot \nabla \Pi_k^\nabla v_h \, \mathrm{d}x$$

but this is true by the definition of Π_k^∇ since $\Pi_k^\nabla v_h \in \mathbb{P}_k(P)$. □

Consistency and Stability

So. . . why we don't take the consistency term and drop the rest?

It's time for a bit of philosophical bla-bla.

The split between a **consistency** term and a **stability** term is one of the main feature of the method, and can be applied also in different contexts $\Longrightarrow$ Hourglass stabilization.

The consistency term is the our intrepid hero: Don Quijote!
He is computed exactly and takes care of the order of approximation of the method (i.e. of the polynomial part of the approximate solution).

But precisely like Don Quijote, he needs support, because if taken alone it would provoke disasters. . . in fact **the global stiffness matrix of the consistency term alone would be singular**.

The kernel of the local stiffness matrices would contain the constant functions - and this is OK, precisely as in the FEM - but also all VEM functions which are in the kernel of the operator Π_k^∇, which are a lot.

Consistency and Stability

In fact:

$$\dim(\ker \Pi_k^\nabla)) = \dim(V_k(P)) - \pi_k = k\,N_e + \pi_{k-2} - \pi_k = k\,(N_e - 2) - 1$$

(remember that $N_e \geq 3$).

Hence the **consistency term** alone is not enough.

The stability term plays the role of "Su inseparable escudero Sancho Panza".

He supports **consistency** by doing the dirty job of making the global stiffness matrix non-singular without upsetting the approximation properties.

The stability term is not computable exactly, but it is enough to approximated it quite roughly – it should only scale in the right way with respect to h.

Consistency and Stability

The basic idea is to observe that $\Pi_k^\nabla \varphi_i$ is a (known) polynomial, so we can compute its degrees of freedom, i.e. we can write $\Pi_k^\nabla \varphi_i$ in the basis $\{\varphi_k\}$:

$$\Pi_k^\nabla \varphi_i = \sum_{|\boldsymbol{\alpha}|=1}^{k} s_i^{\boldsymbol{\alpha}} m_{\boldsymbol{\alpha}} \quad \text{and} \quad m_{\boldsymbol{\alpha}} = \sum_{k=1}^{N^{\text{dof}}} \text{dof}_k(m_{\boldsymbol{\alpha}})\varphi_k$$

hence

$$\Pi_k^\nabla \varphi_i = \sum_{|\boldsymbol{\alpha}|=1}^{k} s_i^{\boldsymbol{\alpha}} \sum_{k=1}^{N^{\text{dof}}} \text{dof}_j(m_{\boldsymbol{\alpha}})\varphi_k = \sum_{k=1}^{N^{\text{dof}}} \pi_i^k \varphi_j$$

Then we have the expansion

$$(I - \Pi_k^\nabla)\varphi_i = \sum_{k=1}^{N^{\text{dof}}} (\delta_{ik} - \pi_i^k)\varphi_k$$

Consistency and Stability

Consequently, the stability term (Sancho Panza) of the stiffness matrix can be written as

$$\int_P \nabla(I-\Pi_k^\nabla)\varphi_j \cdot \nabla(I-\Pi_k^\nabla)\varphi_i \,\mathrm{d}x = \sum_{k,\ell=1}^{N^{\mathrm{dof}}} (\delta_{ik}-\pi_i^k)(\delta_{i\ell}-\pi_i^\ell) \boxed{\int_P \nabla\varphi_k \cdot \nabla\varphi_\ell \,\mathrm{d}x}$$

and we can approximate the boxed term with δ_{kl}:

$$\boxed{\int_P \nabla\varphi_k \cdot \nabla\varphi_\ell \,\mathrm{d}x} \approx \delta_{kl}$$

without loosing accuracy.

In terms of the bilinear form, this amounts to approximate

$$a^P((u_h - \Pi_k^\nabla u_h),(v_h - \Pi_k^\nabla v_h)) \approx \mathcal{S}^P((u_h - \Pi_k^\nabla u_h),(v_h - \Pi_k^\nabla v_h))$$

where $\mathcal{S}^P(\cdot,\cdot)$ is the bilinear form defined by

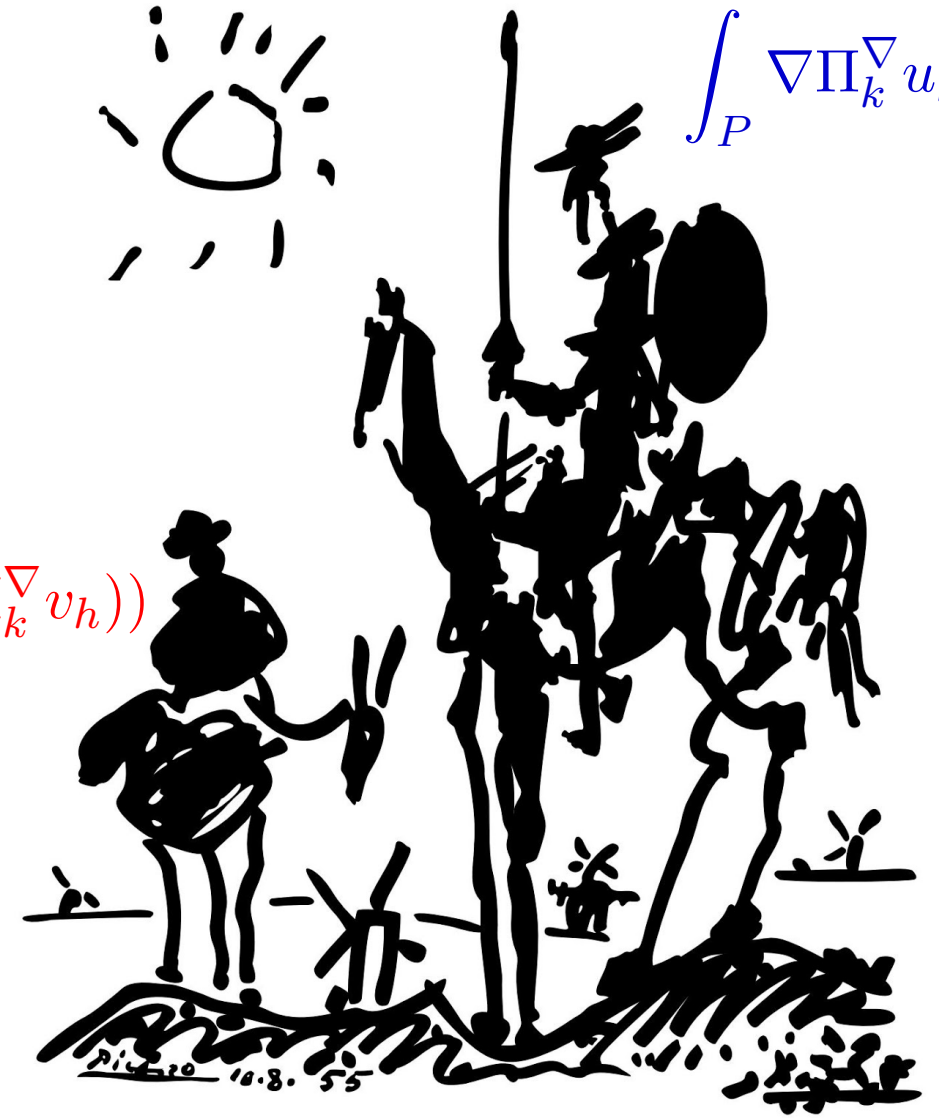
$$\mathcal{S}^P(\varphi_k,\varphi_\ell) = \delta_{kl}$$

Consistency and Stability

$$\boxed{a_h^P(u_h, v_h)}$$

$$\int_P \nabla \Pi_k^\nabla u_h \cdot \nabla \Pi_k^\nabla v_h \, \mathrm{d}x$$

$$\mathcal{S}^P((u_h - \Pi_k^\nabla u_h), (v_h - \Pi_k^\nabla v_h))$$



k-consistency

Local approximate bilinear form:

$$a_h^P(u_h, v_h) = \int_P \nabla \Pi_k^\nabla u_h \cdot \nabla \Pi_k^\nabla v_h \, \mathrm{d}x + \mathcal{S}^P((u_h - \Pi_k^\nabla u_h), (v_h - \Pi_k^\nabla v_h))$$

We are ready now to state one of the fundamental ingredients of the analysis of the Virtual Element Method: the **k-consistency property**.

It says that when one of the two entries is a polynomial, then the local bilinear form is computed exactly:

$$a_h^P(u_h, p_k) = a^P(u_h, p_k)$$

k-consistency

Proof. Since $\Pi_k^\nabla p_k = p_k$, it is clear that the stability term is zero.

Then we have:

$$
\begin{aligned}
a_h^P(u_h, p_k) &= \int_P \nabla \Pi_k^\nabla u_h \cdot \nabla \Pi_k^\nabla p_k \, \mathrm{d}x && \text{since the stability term is zero} \\
&= \int_P \nabla \Pi_k^\nabla u_h \cdot \nabla p_k \, \mathrm{d}x && \text{since } \Pi_k^\nabla p_k = p_k \\
&= \int_P \nabla u_h \cdot \nabla p_k \, \mathrm{d}x && \text{by definition of } \Pi_k^\nabla \\
&= a^P(u_h, p_k)
\end{aligned}
$$

The same property is inherited by the global approximate bilinear form:

$$
a_h(u_h, v_h) = \sum_{P \in \mathcal{P}_h} a_h^P(u_h, v_h)
$$

Patch test

Each Galerkin Method passes the patch test, meaning that if the exact solution is in the discrete space, then the method will find it.

This happens if the discrete bilinear form coincides with the exact one.

For standard Finite Elements, it is enough that all integrations are performed exactly.

For the VEM this is no more the case, since our discrete bilinear form $a_h(\cdot,\cdot)$ is only an approximation of the exact one $a(\cdot,\cdot)$.

However, the k-consistency implies that if the exact solution is a (global) polynomial, then the VEM will find it.

Beware: this is not true if the exact solution is in the VEM space but it is not a global polynomial!!!

For the time being, we assume to compute the VEM right-hand-side without error (we will come back later to this point).

Polynomial Patch test

Assume that the exact solution is a global polynomial $p_k \in \mathbb{P}_k(\Omega)$:

$$a(p_k, v) = F(v) \quad \text{for all } v \in H_0^1(\Omega).$$

If V_h is our VEM space, we have $p_k \in V_h$ so that (patch test)

$$a(p_k, v_h) = F(v_h) \quad \text{for all } v_h \in V_h.$$

But for the k-consistency we have $a(p_k, v_h) = a_h(p_k, v_h)$, hence p_k solves also the discrete problem with the approximate bilinear form:

$$a_h(p_k, v_h) = F(v_h) \quad \text{for all } v_h \in V_h.$$

Note (again) that we have assumed the exact computation of $F(v_h)$. □

Treatment of the right-hand-side

We have to compute

$$F(v_h) := \int_P f v_h \, \mathrm{d}x$$

for $v_h \in V_k(P)$.

It can be shown that we can approximate v_h with its L^2 projection onto the space of polynomials of degree $k-2$:

$$\int_P f v_h \, \mathrm{d}x \approx \int_P f \, \Pi^0_{k-2} v_h \, \mathrm{d}x := F_h(v_h)$$

The polynomial $\Pi^0_{k-2} v_h$ can be computed starting from the degrees of freedom of v_h (remember that the moments up to degree $k-2$ are the internal degrees of freedom).

For $k = 1$ there is an ad-hoc precedure.

A Convergence Result

We report here a Theorem which has been proved in

L. Beirão da Veiga, F. Brezzi, A. Cangiani, G. Manzini, L.D. Marini, A. Russo: *Basic principles of Virtual Element Methods*, Math. Models Methods Appl. Sci. 23, (2013), 199-214.

It is basically the analogous of Céa's Lemma for VEM.

Assume that the local bilinear form $a_h^P(\cdot,\cdot)$ has the following properties:

- *k-consistency*: for all $p_k \in \mathbb{P}_k(P)$ and for all $v_h \in V_k(P)$

$$a_h^P(p_k, v_h) = a^P(p_k, v_h)$$

- *stability*: there exist two positive constants α_* and α^* such that

$$\alpha_* a^P(v_h, v_h) \leq a_h^P(v_h, v_h) \leq a^P(v_h, v_h) \quad \text{for all } v_h \in V_k(P)$$

A Convergence Result

Then we have the following error estimate:

$$|u - u_h|_{1,\Omega} \leq C \left(|u - u_I|_{1,\Omega} + |u - u_\pi|_{h,1,\Omega} + \mathcal{F}_h \right)$$

where:

- u is the exact solution;
- u_h is the VEM solution;
- u_I is any element of V_h (will be used with the interpolant of u in V_h)
- u_π is any piecewise polynomial function with respect to the decomposition (will be the local projection of u) and $|\cdot|_{1,h,\Omega}$ is the broken H^1 seminorm;
- $\mathcal{F}_h$ is the error in the approximation of the right-hand-side, i.e.

$$|F(v_h) - F_h(v_h)| \leq \mathcal{F}_h |v_h|_{1,\Omega}.$$

A Convergence Result

Proof. Take any function $u_I \in V_h$ and set $\delta_h := u_h - u_I$.

$$
\begin{aligned}
\alpha_* |\delta_h|^2_{1,\Omega} &= \alpha_* a(\delta_h, \delta_h) \leq a_h(\delta_h, \delta_h) \quad \text{(stability)} \\
&= a_h(u_h, \delta_h) - a_h(u_I, \delta_h) \quad \text{(definition of } \delta_h\text{)} \\
&= F_h(\delta_h) - \sum_{P \in \mathcal{P}_h} a_h^P(u_I, \delta_h) \quad (u_h \text{ is the discrete solution}) \\
&= F_h(\delta_h) - \sum_{P \in \mathcal{P}_h} \left[a_h^P(u_I - u_\pi, \delta_h) + a_h^P(u_\pi, \delta_h) \right] \quad (\pm u_\pi) \\
&= F_h(\delta_h) - \sum_{P \in \mathcal{P}_h} \left[a_h^P(u_I - u_\pi, \delta_h) + a^P(u_\pi, \delta_h) \right] \quad (k\text{-consistency}) \\
&= F_h(\delta_h) - \sum_{P \in \mathcal{P}_h} \left[a_h^P(u_I - u_\pi, \delta_h) + a^P(u_\pi - u, \delta_h) \right] - a(u, \delta_h) \quad (\pm u) \\
&= F_h(\delta_h) - \sum_{P \in \mathcal{P}_h} \left[a_h^P(u_I - u_\pi, \delta_h) + a^P(u_\pi - u, \delta_h) \right] - F(\delta_h) \quad (u \text{ ex.s.}) \\
&= \left[F_h(\delta_h) - F(\delta_h) \right] - \sum_{P \in \mathcal{P}_h} \left[a_h^P(u_I - u_\pi, \delta_h) + a^P(u_\pi - u, \delta_h) \right]
\end{aligned}
$$

A Convergence Result

Hence we have

$$\alpha_* |\delta_h|^2_{1,\Omega} = [F_h(\delta_h) - F(\delta_h)] - \sum_{P \in \mathcal{P}_h} \left[a_h^P(u_I - u_\pi, \delta_h) + a^P(u_\pi - u, \delta_h) \right]$$

Now:

- $|F(\delta_h) - F_h(\delta_h)| \leq \mathcal{F}_h |\delta_h|_{1,\Omega}$.
- $a^P(\cdot,\cdot)$ is obviously continuous;
- $a_h^P(\cdot,\cdot)$ is also continuous:

$$a_h^P(u_h, v_h) \leq \left(a_h^P(u_h, u_h)\right)^{1/2} \left(a_h^P(v_h, v_h)\right)^{1/2} \leq \alpha^* |u_h|_{1,P} |v_h|_{1,P}$$

hence

$$|\delta_h|_{1,\Omega} \leq C \left(\mathcal{F}_h + |u_I - u_\pi|_{1,h,\Omega} + |u - u_\pi|_{1,h,\Omega} \right)$$

and the theorem follows trivially by the triangle inequality:

$$|u_I - u_\pi|_{1,h,\Omega} \leq |u - u_I|_{1,h,\Omega} + |u - u_\pi|_{1,h,\Omega}.$$

A Convergence Result

The error estimates when h goes to zero follow from Bramble-Hilbert type theorems for polygons (star-shaped domains, Dupont-Scott theory) and standard approximation results for the right-hand-side.

The stability hypothesis means essentially that the stability bilinear form "scales" as the original bilinear form when h tends to zero.

"$\nabla\varphi_i$" behaves like $1/h$ since all degrees of freedom scale in the same way with respect to h so

$$\int_P |\nabla\varphi_j|^2 \,\mathrm{d}x \quad \text{scales like} \quad |P|\,\frac{1}{h^2} \approx 1$$

since we are in two dimensions (in three dimensions would be h). Other stabilizations are possible.